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THE DIFFERENTIABILITY OF NONLINEAR SEMIGROUPS IN UNIFORMLY CONVEX SPACES

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ABSTRACT

Let $T(t)$ be a semigroup on a subset of Banach space X. $T(t)$ is generated by a product integral of the resolvent J_λ of an accretive operator A. If X is a Hilbert space, it is known that for x in the domain of A, $\|J_t x - T(t)x\| = o(t)$ as t decreases to zero. We show this is true when X is uniformly convex, and deduce some consequences.

1. Introduction

If a semigroup $T(t)$ on a subset of a Banach space $(X, \| \quad \|)$ is generated, in the sense of Crandall and Liggett [5], by an accretive operator A, then the backward Euler scheme

$$x_n + \lambda A x_n \ni x_{n-1}, \quad x_0 = x$$

(1.1) or equivalently

$$x_n = J_\lambda^n x, \quad J_\lambda = (I + \lambda A)^{-1}$$

converges to $T(t)x$. That is to say $J_\lambda^n x \to T(t)x$ as $\lambda \downarrow 0$, $n\lambda \to t$.

For a 'local proof' of convergence, one would require at very least

$$\|J_t x - T(t)x\| = o(t) \qquad \text{as } t \downarrow 0. \tag{1.2}$$

However, accretiveness is a global estimate, and the convergence proof uses this fact. Indeed the best that one obtains from [5] is

$$\|J_t x - T(t)x\| \leqq K_x t. \tag{1.3}$$

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It is known that this is as much as one can expect in general Banach spaces. A best possible value for K_x is found later ($K_x = (2/e)|Ax|$). Thus, bad local behaviour is globally smoothed. The situation is different in a Hilbert space: (1.2) holds.

Accretiveness is a metric property, and [5] uses metric estimates. Before this, uniform convexity of the dual of X was commonly assumed. This was to ensure the duality map be sufficiently well behaved; [7, lemma 1.2], for example. The methods involved a somewhat curious mixture of topology and estimation.

The main result we prove is that (1.2) holds if X is uniformly convex. The idea of the proof is geometrical. The beautiful triangle inequality of Clarkson is used to show the 'angle' between two vectors is small. This, together with the fact that the vectors have about the same length, is enough to show their difference is small.

Incidental to the proof of the above, we obtain a characterization of the infinitesimal generator of the semigroup $T(t)$. This generator, and its relation to the corresponding initial value problem, is investigated in the sequel. We obtain some modest improvements of those results of Miyadera in [10] which specifically relate to uniformly convex spaces. The results themselves are somewhat predictable; our interest is more in the methodology employed in arriving at them. As already mentioned, the infinitesimal generator of $T(t)$ is obtained 'for free'. Our procedure is to relate this generator to the given operator A, and to its canonical restriction. In contrast, [10] assumes conditions which ensure the canonical restriction is well behaved, and then shows it is the infinitesimal generator.

QUESTION. The Crandall–Liggett paper gives the estimate

$$\| J_\lambda^m x - T(m\lambda)x \| \leqq O(\lambda^{1/2})$$

as $\lambda \downarrow 0$ and $m\lambda$ bounded. Can the exponent of λ be increased in the case X is a Hilbert space?

2. Preliminaries

$(X, \| \quad \|)$ denotes a real Banach space. Its *modulus of convexity* δ is defined by

$$\delta(\varepsilon) = \inf\{1 - \|a + b\|/2 \colon \|a\| = \|b\| = 1, \|a - b\| = \varepsilon\}$$

for $0 \leqq \varepsilon \leqq 2$. X is *uniformly convex* if $\delta(\varepsilon) > 0$ for $\varepsilon > 0$. It is shown in [6, corollary 5] that δ is nondecreasing. Hence, if X is uniformly convex and $\delta(\varepsilon_n) \to 0$, then $\varepsilon_n \to 0$.

Following Clarkson [2], the *generalized angle* $\alpha(a,b)$ between two non-zero vectors a, b is defined by

$$\alpha(a,b)=\left\|\frac{a}{\|a\|}-\frac{b}{\|b\|}\right\|.$$

It is easily verified that

$$\big|\,\|a\|\alpha(a,b)-\|a-b\|\,\big|\leqq\big|\,\|a\|-\|b\|\,\big|. \tag{2.1}$$

As a special case of Clarkson's triangle inequality [2, theorem 3] we have

$$\|a+b\|\leqq\{1-2\delta(\alpha(a+b,a))\}\|a\|+\|b\| \tag{2.2}$$

whenever the right-hand side is defined.

If $\Omega\subset X$ then $T\colon[0,\infty)\times\Omega\to\Omega$ is a (c_0-)*semigroup* on Ω if (i) $T(0)x=x$, (ii) $T(t+s)x=T(t)T(s)x$, (iii) $T(t)x$ is strongly t-continuous. If, in addition, $T(t)$ is a contraction on Ω for each t, then T is a *contraction* semigroup.

Following [4, definition 1.4], we have

$$[x,y]_\lambda=(\|x+\lambda y\|-\|x\|)/\lambda\downarrow[x,y]_+\qquad\text{as }\lambda\downarrow 0.$$

A set valued operator $A\subset X\times X$ is *accretive* if

$$[x-y,x'-y']_+\geqq 0,\qquad x'\in Ax,\quad y'\in Ay. \tag{2.3}$$

If the backward Euler scheme (1.1) is to exist for $x\in D(A)=\{x\colon Ax\neq\emptyset\}$, we must have the range condition

$$R(I+\lambda A)\supset D(A) \tag{2.4}$$

for sufficiently small $\lambda>0$. The Crandall–Liggett theorem then asserts (1.1) converges to a contraction semigroup $T(t)$ on the closure of $D(A)$. (The extension from $D(A)$ to its closure is by continuity.) With these assumptions, if $x\in D(A)$ and t decreases to zero

$$\|x-J_tx\|/t\uparrow|Ax|\leqq\inf\{\|y\|\colon y\in Ax\}, \tag{2.5}$$

$$\|x-T(t)x\|/t\overset{\rightarrow}{\leqq}|Ax|. \tag{2.6}$$

For $z'\in Az$, $s<t$

$$\|z-T(t)x\|-\|z-T(s)x\|\leqq\int_s^t[z-T(\tau)x,z']_+d\tau. \tag{2.7}$$

The first result is elementary and defines $|Ax|$. Both (2.5) and (2.6) are taken from [3], or [11]. (2.7) is the condition that $T(t)x$ is an *integral solution* [1] of $x' + Ax \ni 0$. Sketch proof:

Since A is accretive and $(J_\lambda^{k-1}x - J_\lambda^k x)/\lambda \in AJ_\lambda^k x$,

$$[z - J_\lambda^k x, z' - (J_\lambda^{k-1}x - J_\lambda^k x)/\lambda]_\lambda \geqq 0.$$

This rearranges to become

$$\| z - J_\lambda^k x \| - \| z - J_\lambda^{k-1} x \| \leqq \lambda [z - J_\lambda^{k-1}x, z']_\lambda.$$

Summing $k = m+1, \cdots, n$, and making a change of variables

$$\| z - J_\lambda^n x \| - \| z - J_\lambda^m x \| \leqq \int_{m\lambda}^{n\lambda} [z - J_\lambda^{[\tau/\lambda]}x, z']_\lambda d\tau$$

($[\tau/\lambda]$ denotes the integer part of τ/λ).

Now let $\lambda \downarrow 0$, $n\lambda \to t$, $m\lambda \to s$, and take limsup on both sides. Apply Fatou's Lemma to the integral, and note that limsup of the integrand is majorized by $[z - T(\tau)x, z']_\varepsilon$ for any $\varepsilon > 0$. Apply monotone convergence as $\varepsilon \downarrow 0$.

An equivalent, and more convenient, differential version of (2.7) is

$$D\| z - T(s)x \| \leqq [z - T(s)x, z']_+, \qquad z' \in Az. \tag{2.8}$$

Here D denotes any one of the four Dini derivatives with respect to s. The substitutions $z = J_\lambda x$, $z' = (x - J_\lambda x)/\lambda$ lead to a number of interesting estimates. We mention only two.

$$\begin{aligned} D\| J_\lambda x - T(s)x \| &\leqq [J_\lambda x - T(s)x, (x - J_\lambda x)/\lambda]_\lambda \\ &= \frac{1}{\lambda}\{\| x - T(s)x \| - \| J_\lambda x - T(s)x \|\} \\ &\leqq \frac{1}{\lambda}\{2\| x - T(s)x \| - \| x - J_\lambda x \|\}. \end{aligned}$$

The first inequality integrates to give

$$\| J_\lambda x - T(t)x \| \leqq e^{-t/\lambda}\| x - J_\lambda x \| + \frac{1}{\lambda}\int_0^t e^{(s-t)/\lambda}\| x - T(s)x \| ds. \tag{2.9}$$

The second

$$\| J_\lambda x - T(t)x \| \leqq (1 - t/\lambda)\| x - J_\lambda x \| + \frac{2}{\lambda}\int_0^t \| x - T(s)x \| ds. \tag{2.10}$$

Set $\lambda = t$ and use (2.6) to estimate the integrals. Then, $\| J_t x - T(t)x \| / t$ is majorized by $(2/e)| Ax |$ in (2.9), and by $| Ax |$ in (2.10). Later we give an example where the constant $2/e$ is attained.

If X is a Hilbert space it is natural to work with the square of the norm. Note that $(x, y) = \|x\| [x, y]_+$. With the same substitutions for z and z', (2.8) becomes

$$D\|J_\lambda x - T(s)x\|^2 \leqq \frac{2}{\lambda}(J_\lambda x - T(s)x, x - J_\lambda x)$$

$$= \frac{1}{\lambda}\{\|x - T(s)x\|^2 - \|J_\lambda x - T(s)x\|^2 - \|x - J_\lambda x\|^2\}.$$

This differential inequality integrates to give

$$\text{(2.11)} \qquad \|J_\lambda x - T(t)x\|^2 \leqq (2e^{-t/\lambda} - 1)\|x - J_\lambda x\|^2 + \frac{1}{\lambda}\int_0^t e^{(s-t)/\lambda}\|x - T(s)x\|^2 ds.$$

Estimating as before

$$\|J_t x - T(t)x\|/t \leqq (1 - 2/e)^{1/2}\{|Ax|^2 - (\|x - J_t x\|/t)^2\}^{1/2}$$

$$= o(1) \qquad \text{as } t \downarrow 0.$$

Estimate (2.10) has been used in interpolation theory [11]. On the other hand, the stronger (2.9) may be new. It, together with (2.11), was discovered by the author using a quite different argument involving the Poisson convergence theorem for Bernoulli trials.

3. Uniformly convex spaces

We have seen that (2.9) and (2.10) do not directly give the 'little oh' estimate (1.2). If (1.2) does hold, one would roughly expect the function $F(t) = \|J_\lambda x - T(t)x\|$ to decrease with speed $|Ax|$ when $0 < t \ll \lambda \ll 1$. The derivatives of the right-hand sides of (2.9) and (2.10) with respect to t are both equal to $-\|x - J_\lambda x\|/\lambda \simeq -|Ax|$ at $t = 0$. Thus (2.9), (2.10) are consistent with (1.2) for $t \ll \lambda$. It was this observation which led to the technique that follows.

We assume A is an accretive operator satisfying the range condition (2.4), and $T(t)$ the semigroup it generates.

THEOREM 1. *If X is uniformly convex and $x \in D(A)$ then*

$$\|J_t x - T(t)x\| = o(t) \qquad \text{as } t \downarrow 0.$$

Theorem 1 follows from

THEOREM 2. *If X is uniformly convex and $x \in D(A)$, the limits as $t \downarrow 0$ of $(x - J_t x)/t$ and $(x - T(t)x)/t$ both exist and are equal.*

PROOF. Since the norms of the above expressions both converge to $|Ax|$, it suffices to assume $|Ax| \neq 0$. The proof is in two parts.

Step 1. The first limit exists.

For $0 < s < t$, let $\alpha_{s,t}$ denote $\alpha(x - J_s x, x - J_t x)$. By the triangle inequality (2.2)

$$\|x - J_t x\| \leqq \{1 - 2\delta(\alpha_{s,t})\}\|x - J_s x\| + \|J_t x - J_s x\|.$$

Using the resolvent identity [5, lemma 1.2] on the last term

$$\|J_t x - J_s x\| = \left\| J_s\left(\frac{s}{t}x + \frac{t-s}{t}J_t x\right) - J_s x\right\|$$
$$\leqq \left(1 - \frac{s}{t}\right)\|x - J_t x\|.$$

The triangle inequality then rearranges to become

$$2\delta(\alpha_{s,t})\|x - J_s x\|/s \leqq \|x - J_s x\|/s - \|x - J_t x\|/t. \tag{3.1}$$

Hence

$$\limsup 2\delta(\alpha_{s,t})|Ax| \leqq |Ax| - |Ax| = 0$$

where the limsup is taken as $t \downarrow 0, 0 < s < t$. Consequently $\alpha_{s,t} \to 0$.

Now set $a = (x - J_t x)/t$, $b = (x - J_s x)/s$ in (2.1). It follows that

$$\|(x - J_t x)/t - (x - J_s x)/s\| \to 0 \qquad \text{as } t \downarrow 0, \quad 0 < s < t,$$

so, by completeness, $(x - J_t x)/t$ converges.

Step 2. We show

$$\|(x - J_t x)/t - (x - T(s)x)/s\| \to 0 \qquad \text{as } t, s/t \to 0.$$

This proves existence of the second limit and its equality with the first. In view of (2.1) it suffices to show $\alpha_{s,t} = \alpha(x - T(s)x, x - J_t x) \to 0$ as t and s/t decrease to zero.

Using the triangle inequality (2.2)

$$\|x - J_t x\| \leqq \{1 - 2\delta(\alpha_{s,t})\}\|x - T(s)x\| + \|J_t x - T(s)x\|.$$

The last term is estimated by (2.10). ((2.9) would do equally well.)

$$\|J_t x - T(s)x\| \leqq (1 - s/t)\|x - J_t x\| + (s^2/t)|Ax|.$$

The triangle inequality then rearranges to become

$$2\delta(\alpha_{s,t})\|x - T(s)x\|/s \leqq \|x - T(s)x\|/s - \|x - J_t x\|/t + (s/t)|Ax|.$$

Take limits, and the proof is complete.

REMARK. A similar argument to that used in Step 1 would show $(x - T(t)x)/t$ is convergent as $t \downarrow 0$. This is proved in [10, theorem 3] under the additional assumptions that A is maximal accretive and its canonical restriction is single valued. (Maximality is not a problem, since A may be replaced by a maximal extension.) If this result is assumed, Step 1 is redundant.

A similar method to the above gives results on left differentiability of $T(t)x$. An additional complication is that $|AT(t)x|$ may not be left continuous. (It is right continuous and nonincreasing.)

THEOREM 3. *If X is uniformly convex, $T(t)x \in D(A)$ for $t = t_0 > 0$ and $|AT(t)x|$ is continuous at t_0, then $T(t)x$ is strongly differentiable at t_0.*

PROOF. Let $0 < t < t_0$ and $\alpha_{\lambda,t} = \alpha(J_\lambda T(t_0)x - T(t)x, T(t_0)x - T(t)x)$. By the triangle inequality (2.2)

$$\|J_\lambda T(t_0)x - T(t)x\| - \|J_\lambda T(t_0)x - T(t_0)x\| \leqq \{1 - 2\delta(\alpha_{\lambda,t})\}\|T(t_0)x - T(t)x\|.$$

To estimate the left-hand side, set $z = J_\lambda T(t_0)x$, $z' = (T(t_0)x - J_\lambda T(t_0)x)/\lambda$ in (2.8) and estimate in the same way as (2.10) was obtained.

$$D\|J_\lambda T(t_0)x - T(s)x\| \leqq \frac{1}{\lambda}\{2\|(t_0)x - T(s)x\| - \|J_\lambda T(t_0)x - T(t_0)x\|\}.$$

Integrate from t to t_0.

$$\begin{aligned}&\|J_\lambda T(t_0)x - T(t_0)x\| - \|J_\lambda T(t_0)x - T(t)x\| \\ &\leqq \frac{2}{\lambda}\int_t^{t_0} \|T(t_0)x - T(s)x\|ds - \frac{t_0 - t}{\lambda}\|J_\lambda T(t_0)x - T(t_0)x\|.\end{aligned}$$

Hence

$$\begin{aligned}2\delta(\alpha_{\lambda,t})\|T(t_0)x - T(t)x\|/(t_0 - t) &\leqq \|T(t_0)x - T(t)x\|/(t_0 - t) \\ &- \|J_\lambda T(t_0)x - T(t_0)x\|/\lambda + \frac{2}{\lambda(t_0 - t)}\int_t^{t_0}\|T(t_0)x - T(s)x\|ds.\end{aligned} \tag{3.2}$$

Now, if $t \leqq s < t_0$

$$\begin{aligned}\|T(t_0 + t_0 - s)x - T(t_0)x\|/(t_0 - s) &\leqq \|T(t_0)x - T(s)x\|/(t_0 - s) \\ &\leqq |AT(t)x|.\end{aligned} \tag{3.3}$$

As $t \uparrow t_0$, both sides converge to $|AT(t_0)x|$. If $|AT(t_0)x| = 0$ the theorem is proved. In any case the right-hand side of (3.2) converges to zero as $\lambda \downarrow 0$ and $(t_0 - t)/\lambda \to 0$. Hence, if $|AT(t_0)x| \neq 0$, $\alpha_{\lambda,t} \to 0$.

Set $a = (J_\lambda T(t_0)x - T(t)x)/\lambda$, $b = (T(t_0)x - T(t)x)/(t_0 - t)$ in (2.1), and let $\lambda \downarrow 0$, $(t_0 - t)/\lambda \to 0$. Note that

$$a = \frac{t_0 - t}{\lambda}(T(t_0)x - T(t)x)/(t_0 - t) + (J_\lambda T(t_0)x - T(t_0)x)/\lambda$$

$$\to 0 + \frac{d^+}{dt} T(t_0)x$$

by (3.3) and Theorem 2. Also $\| a \|$ and $\| b \|$ both converge to $| AT(t_0)x |$. Hence

$$\left\| \frac{d^+}{dt} T(t_0)x - (T(t_0)x - T(t)x)/(t_0 - t) \right\| \to 0$$

as $t \uparrow t_0$. Thus, $T(t)x$ has a left derivative, which equals its right derivative.

In order to characterize the infinitesimal generator of $T(t)$, we need to extend definition (2.5) of $|Ax|$ to the closure of $D(A)$. A more than adequate assumption to achieve this is

$$R(I + \lambda A) \supset \overline{D(A)} \tag{3.4}$$

for sufficiently small $\lambda > 0$. This is the range condition assumed in [5]. The *generalized semigroup domain*

$$\hat{D}(A) = \{x \in \overline{D(A)} : |Ax| < \infty\}$$

has the pleasing property of being forward flow-invariant for the semigroup $T(t)$. Also (2.6) holds for $x \in \hat{D}(A)$. All the above statements are proved in both [3] and [11]. We observe that the proofs of Theorems 1, 2 and 3 remain unchanged if $D(A)$ is replaced by $\hat{D}(A)$ in their statements. In particular, for $x \in \hat{D}(A)$, denote the common limit in Theorem 2 by A^*x.

THEOREM 4. *Let X be uniformly convex,* (3.4) *hold, and A^*x denote the limit in Theorem* 2. *Then*

(1) *$A^*: \hat{D}(A) \to X$, A^* single valued.*

(2) *$A \cup A^*$ is accretive.*

(3) $\| A^*x \| = |Ax|$.

(4) *$-A^*$ is the strong infinitesimal generator of $T(t)$.*

(5) *$A^*T(t)x$ is right continuous for $x \in \hat{D}(A)$.*

(6) *If $|AT(t)x|$ is left continuous at $t = t_0 > 0$ then $A^*T(t)$ is continuous at t_0 and*

$$\frac{d}{dt} T(t)x + A^*T(t)x = 0 \qquad \text{at } t = t_0. \tag{3.5}$$

PROOF. The proofs of (1), (2), (3) are routine, and are omitted.

If $x \notin \hat{D}(A)$ then $\|x - T(t)x\|/t$ is unbounded as t decreases to zero, so x is not in the domain of the infinitesimal generator of $T(t)$. On the other hand, if $x \in \hat{D}(A)$, then $(T(t)x - x)/t \to -A^*x$.

To prove (5) we first require a lemma, which, in a certain sense, says that $(x - J_\lambda x)/\lambda$ converges uniformly to A^*x.

LEMMA 5. *For $\varepsilon > 0$ there exists $\delta > 0$ such that if $x \in \hat{D}(A)$, $|Ax| \neq 0$ then*

$$1 - \frac{\|x - J_\lambda x\|}{\lambda |Ax|} < \delta \quad \textit{implies} \quad \left\| A^*x - \frac{x - J_\lambda x}{\lambda} \right\| < \varepsilon |Ax|.$$

PROOF. From (2.1) we have

$$\begin{aligned}\|A^*x - (x - J_\lambda x)/\lambda\| &\leqq |Ax| \alpha(A^*x, x - J_\lambda x) + |Ax| - \|x - J_\lambda x\|/\lambda \\ &\leqq |Ax| \{\alpha(A^*x, x - J_\lambda x) + \delta\}.\end{aligned}$$

Set $t = \lambda$ and let s decrease to zero in (3.1) to get $2\delta(\alpha(A^*x, x - J_\lambda x)) \leqq \delta$. Hence, by choosing δ sufficiently small, we make $\alpha(A^*x, x - J_\lambda x)$, and also the right-hand side of the above inequality, as small as we please. The Lemma is proved.

PROOF OF (5). Let $x \in \hat{D}(A)$. If $|Ax| = 0$ then $A^*T(s)x = 0$, so we may assume $|Ax| > 0$.

Choose $\varepsilon > 0$. Then choose the δ in Lemma 5. Finally choose $\tau > 0$ such that $|AT(s)x| > (1 - \delta)|Ax|$ for $0 \leqq s \leqq \tau$. Let

$$U_\lambda = \left\{ s \in [0, \tau] : 1 - \frac{\|T(s)x - J_\lambda T(s)x\|}{\lambda |Ax|} < \delta \right\}.$$

Then, as λ decreases to zero, $\{U_\lambda\}$ is a nondecreasing family of open subsets of $[0, \tau]$. Moreover, the expression in the definition of U_λ converges to $1 - |AT(s)x|/|Ax| < \delta$. $\{U_\lambda\}$ is an open covering of $[0, \tau]$, so for some $\lambda > 0$, $U_\lambda = [0, \tau]$. With this choice of λ

$$1 - \frac{\|T(s)x - J_\lambda T(s)x\|}{\lambda |AT(s)x|} < \delta.$$

By Lemma 5, for $0 \leqq s \leqq \tau$

$$\|A^*T(s)x - (T(s)x - J_\lambda T(s)x)/\lambda\| \leqq \varepsilon |AT(s)x| \leqq \varepsilon |Ax|.$$

Thus

$$\|A^*T(s)x - A^*x\| \leqq \|(T(s)x - J_\lambda T(s)x)/\lambda - (x - J_\lambda x)/\lambda\| + 2\varepsilon |Ax|.$$

Let s decrease to zero, and take the limit superior. The first term on the right vanishes, the second is as small as we please. This proves (5).

PROOF OF (6). From Theorem 3 we already know that $T(t)x$ is differentiable at t_0, so (3.5) holds by part (4). The proof that $A^*T(t)x$ is left continuous is much the same as that for right continuity, and goes as follows.

We may assume $|AT(t_0)x| \neq 0$, since otherwise $\|A^*T(s)x\| \to 0$ as $s \to t_0$. Having chosen ε and δ as above, we choose $\tau < t_0$ such that $|AT(s)x| > (1-\delta)|AT(\tau)x|$ for $\tau \leqq s \leqq t_0$, and let

$$U_\lambda = \left\{ s \in [\tau, t_0]: 1 - \frac{\|T(s)x - J_\lambda T(s)x\|}{\lambda |AT(\tau)x|} < \delta \right\}.$$

Exactly as before, we deduce

$$\|A^*T(s) - (T(s)x - J_\lambda T(s)x)/\lambda\| \leqq \varepsilon |AT(\tau)x|$$

for some $\lambda > 0$ and $\tau \leqq s \leqq t_0$. Hence, as $s \uparrow t_0$,

$$\limsup \|A^*T(s)x - A^*T(t_0)x\| \leqq 2\varepsilon\, |AT(\tau)x|.$$

The proof of Theorem 4 is complete.

If Theorem 4 is compared with [10, theorem 3] it will be seen that A^* plays the rôle commonly reserved for the *canonical restriction* A^0 of A.

$$A^0x = \{y \in Ax : \|y\| = \inf\{\|z\| : z \in Ax\}\}.$$

A^0 is awkward to work with for the following reasons. (i) A^0 can be multivalued. (ii) $D(A^0)$ may be a proper subset of $D(A)$, even empty. (iii) $D(A^0)$ need not be flow-invariant for the semigroup $T(t)$. The first difficulty is avoided if X and X^* are strictly convex and A is maximal accretive on $D(A)$. If, in addition, X is reflexive then $D(A^0) = D(A)$. [8, lemma 3.10]. To ensure $D(A)$ is flow-invariant we need $\hat{D}(A) = D(A)$. This will be the case if A is maximal accretive on $\overline{D(A)}$ or, more generally, if A is almost demiclosed (i.e. $x_n \to x$, $y_n \in Ax_n$, y_n weakly convergent implies $x \in D(A)$). [3, theorem 2]. None of these assumptions requires that X be uniformly convex. However, if X is uniformly convex, we obtain simple criteria.

THEOREM 5. *Let X be uniformly convex and the range condition* (2.4) *hold. Then, if A is closed, $A^* \subset A^0 \subset A$ and, in particular, $D(A^0) = D(A) = \hat{D}(A)$. If, in addition, X^* is strictly convex then $A^* = A^0$.*

PROOF. Since A is closed, the range condition (3.4) holds, and A^* is defined on $\hat{D}(A)$. The logic goes: A closed $\Rightarrow I + \lambda A$ closed $\Rightarrow J_\lambda$ closed $\Rightarrow D(J_\lambda)$ closed, since J_λ continuous.

Let $x \in \hat{D}(A)$. Then $(x - J_\lambda x)/\lambda \in AJ_\lambda x$. Let λ decrease to zero. The left-hand side converges to A^*x. Using the fact that $J_\lambda x \to x$ and A is closed, $A^*x \in Ax$. By Theorem 4 (3) and (2.5)

$$\|A^*x\| = |Ax| \leqq \inf\{\|y\|: y \in Ax\}.$$

The infimum being achieved by A^*x, $A^*x \in A^0x$. This proves the first part.

Now assume X^* is strictly convex. Let $\tilde{A}$ be a maximal accretive extension of A on $D(A)$. By [8, lemma 3.10], $\tilde{A}^0$ is defined and single valued on $D(A)$. Let $y \in A^0x$. We show $y = A^*x$. We have

$$\|y\| = \|A^*x\| = |Ax| = |\tilde{A}x| \leqq \|\tilde{A}^0x\|.$$

Taken in the order shown, the proofs are: (i) $A^*x \in A^0x$. (ii) Theorem 4 (3). (iii) $|Ax|$ and $|\tilde{A}x|$ are defined by the same resolvent operator (or semigroup). (iv) (2.5). Now $y \in \tilde{A}x$. Therefore $y = \tilde{A}^0x$. Since $A^*x \in A^0x$, $y = A^*x$ as required.

REMARK. It is known that if small order perturbations are allowed in (1.1), then the Nagumo type range condition

$$\text{(3.6)} \qquad \liminf d(R(I + \lambda A), x)/\lambda = 0, \qquad \lambda \downarrow 0, \quad x \in \overline{D(A)}$$

is sufficient to ensure (1.1) converges to a contraction semigroup. See [9] and the references therein. Range condition (3.4) was assumed in order to obtain the generalized semigroup domain. A modified procedure using (3.6) shows $\|x - T(t)x\|/t$ and $\|x - J_{t_n}x_n\|/t_n$ converge to the same limit, $|Ax|$, when $t \downarrow 0$, $t_n \downarrow 0$, $x_n \in D(J_{t_n})$ and $\|x - x_n\|/t_n \to 0$. When $x \in \overline{D(A)}$, the existence of such a sequence $\{t_n, x_n\}$ is guaranteed by (3.6). Because J_t need no longer be defined on $D(A)$, Theorems 1 and 2 require restatement. The additional complication of extra perturbation terms somewhat obscures the essential arguments contained in the proof of Theorem 2. However, the following is easily proved by the previous method.

THEOREM 6. *Let A be accretive and satisfy range condition* (3.6). *Let X be uniformly convex and $x \in \hat{D}(A)$. Then*

$$(x - T(t)x)/t \quad \textit{and} \quad (x - J_{t_n}x_n)/t_n$$

converge to the same limit as $t \downarrow 0$, and $\{t_n, x_n\}$ is any sequence such that $t_n \downarrow 0$, $x_n \in D(J_{t_n})$ and $\|x - x_n\|/t_n \to 0$.

4. Example

The following example is designed to show the constant $(2/e)|Ax|$ is best possible value for K_x in (1.3).

Take $X = C([-1,0])$, $D(A) = \{\phi: \phi, \phi' \in X, \phi'(0) = 1\}$, $A\phi = -\phi'$. It is shown in [12] that A is densely defined, m-accretive and that

$$J_t\psi(s) = e^{s/t}\left(\psi(0) + t + \frac{1}{t}\int_s^0 e^{-\tau/t}\psi(\tau)d\tau\right).$$

Taking $\psi(s) = -s$,

$$J_t\psi(s) = 2te^{s/t} - s - t.$$

Hence $\|\psi - J_t\psi\| = t$, so $|A\psi| = 1$.

Now, since $T(t)\psi$ generates 'segments' of the solution of $x' = 1$, $x(0) = 0$, $T(t)\psi(s) = |s + t|$. Then for $0 < t \leqq 1$

$$\|J_t\psi - T(t)\psi\| \geqq |J_t\psi(-t) - T(t)\psi(-t)| = 2t/e = (2/e)|A\psi|t.$$

The reverse inequality was proved in the paragraph following (2.10). To conclude, we note that $T(t)\psi \in D(A)$ if, and only if, $t = 0$ or $t \geqq 1$.

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